Complétion Jointe de Matrices

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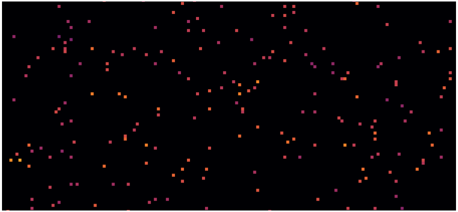
Matrix completion is ...



- **Task:** given a partially observed data matrix $\boldsymbol{X}$, predict the unobserved entries
- Large matrices: # rows, # columns $\approx 10^5, 10^6$.
- Very under-determined (often only 1-2% observed)
- Application to recommender systems, system identification, image processing, microarray data, etc.

Motivations: recommendation systems, Netflix prize

- A popular example is the Netflix challenge (2006-2009)

- Dataset: $480K$ users, $18K$ movies, $100M$ ratings
- Only 1.1% of the matrix is filled!

Some issues ...

- In general, we cannot infer missing ratings without any other information.
- This problem is under-determined, more unknown than observations.
- **Low-rank assumption**: fill matrix such that rank is minimum.
  $\rightarrow$ A few factors explain most of the data.

Completion via rank minimization

$$\text{minimize}_{\boldsymbol{W}} \operatorname{rank}(\boldsymbol{W}) \;\text{ s. t. }\; W_{ij} = \underbrace{X_{ij}}_{\text{observed entries}}, \;(i,j) \in \underbrace{\Omega}_{\text{sampling set}}.$$

- Non-convex problem and combinatorially NP-hard!!

Convex formulation of the rank minimization problem

$$\mathrm{rank}(\boldsymbol{X}) = \sum_{i=1}^{\min\dim(\boldsymbol{X})} \mathbb{1}_{(\sigma_i(\boldsymbol{X})>0)} = \|\sigma(\boldsymbol{X})\|_0.$$

Replace ℓ_0 by ℓ_1 [Fazel (2002), Srebro et al. (2005); Candes and Tao (2010); Recht et al. (2010); Negahban and Wainwright (2011); Klopp (2014)]:

$$\|\boldsymbol{X}\|_* = \sum_{i=1}^{\min\dim(\boldsymbol{X})} (\sigma_i(\boldsymbol{X})).$$

Hence temping to consider

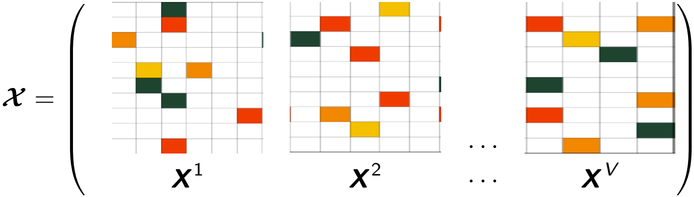
Nuclear norm minimization:

$$\mathrm{minimize}_{\boldsymbol{W}} \|\boldsymbol{W}\|_* \text{ s. t. } W_{ij} = \underbrace{X_{ij}}_{\text{observed entries}}, \ (i,j) \in \underbrace{\Omega}_{\text{sampling set}}.$$

This is a convex problem !

Collective matrix completion: motivations

- Data is often obtained from a collection of matrices $\mathcal{X} = (\boldsymbol{X}^1, \ldots, \boldsymbol{X}^V)$.



- It may be beneficial to leverage all the available user data by various sources.
- **Cold-Start** problem: in recommender systems, when a new user has no rating it is impossible to predict his ratings.
- Shared structure among the sources can be useful to get better predictions.

Collective matrix completion: model setup

- Each source view $\boldsymbol{X}^v \in \mathbb{R}^{d_u \times d_v}$ and $D = \sum_{v=1}^{V} d_v$.
- We assume that the distribution of for each source $\boldsymbol{X}^v$ depends on the matrix of parameters $\boldsymbol{M}^v$.
- **Model:** let B_{ij}^v be independent Bernoulli random variables and independent from X_{ij}^v, with parameter π_{ij}^v.

$$Y_{ij}^v = B_{ij}^v X_{ij}^v.$$

- We can think of the B_{ij}^v as masked variables.
- $\pi_{ij}^v =$ probability to observe the (i, j)-th entry of the v-th source.

Collective matrix completion: sampling scheme

- We consider **general sampling model** where we only assume:

Assumption 1: There exists a positive constant $0 < p < 1$ s.t. $\min_{v \in [V]} \min_{(i,j) \in [d_u] \times [d_v]} \pi^v_{ij} \geq p$.

[Klopp (2015); Klopp et al. (2015)]

- $\pi^v_{i\bullet} = \sum_{j=1}^{d_v} \pi^v_{ij}$ the probability to observe an element from the i-th row of $\boldsymbol{X}^v$.
- $\pi^v_{\bullet j} = \sum_{i=1}^{d_u} \pi^v_{ij}$ the probability to observe an element from the j-th column of $\boldsymbol{X}^v$.
- Let $\pi_{i\bullet} = \max_{v \in [V]} \pi^v_{i\bullet}$, $\pi_{\bullet j} = \sum_{v=1}^{V} \pi_{i\bullet}$, and

$$\max_{(i,j) \in [d_u] \times [d_v]} (\pi_{i\bullet}, \pi_{\bullet j}) \leq \mu.$$

Exponential family noise

- Heterogeneous sources: (ratings), (counting: number of clicks) (binomial: like/dislike)
- General framework: natural exponential family

$$X_{ij}^v | M_{ij}^v \sim h^v(X_{ij}^v) \exp\left(X_{ij}^v M_{ij}^v - G^v(M_{ij}^v)\right).$$

[Gunasekar et al. (2014); Cao and Xie (2016); Lafond (2015)]

- Many distributions belong to the exponential family: Gaussian, binomial, Poisson, exponential, etc.

Assumption 2: - The distribution of X_{ij}^v has sub-exponential tail.
- Strong convexity of the log-partition function G^v.

Exponential family noise: estimation procedure

- Given observations $\boldsymbol{\mathcal{Y}} = (\boldsymbol{Y}^1, \ldots, \boldsymbol{Y}^V)$, we write the negative log-likelihood as

$$\mathscr{L}_{\boldsymbol{\mathcal{Y}}}(\boldsymbol{\mathcal{W}}) = -\frac{1}{d_u D} \sum_{v \in [V]} \sum_{(i,j) \in [d_u] \times [d_v]} B_{ij}^v \big(Y_{ij}^v W_{ij}^v - G^v(W_{ij}^v) \big).$$

- The nuclear norm penalized estimator $\widehat{\boldsymbol{\mathcal{M}}}$ of $\boldsymbol{\mathcal{M}}$ is defined as follows:

$$\widehat{\boldsymbol{\mathcal{M}}} = (\widehat{\boldsymbol{M}}^1, \ldots, \widehat{\boldsymbol{M}}^V) = \operatorname*{argmin}_{\|\boldsymbol{\mathcal{W}}\|_\infty \leq \gamma} \mathscr{L}_{\boldsymbol{\mathcal{Y}}}(\boldsymbol{\mathcal{W}}) + \lambda \|\boldsymbol{\mathcal{W}}\|_*,$$

- $\lambda > 0$ is a positive regularization parameter that balances the trade-off between model fit and privileging a low-rank solution.

Exponential family noise: theoretical guarantee

Theorem [A., Klopp 2018]

Assume that Assumptions 1 and 2 hold and

$$\lambda \approx \frac{\sqrt{\mu} + (\log(d_u \vee D))^{3/2}}{d_u D}.$$

Then, with high probability, one has

$$\frac{1}{d_u D}\|\widehat{\mathcal{M}} - \mathcal{M}\|_F^2 \lesssim \frac{\mathrm{rank}(\mathcal{M})(\mu + (\log(d_u \vee D))^{3/2})}{p^2 d_u D}.$$

Exponential family noise: theoretical guarantee

- **Uniform sampling:** If $c_1/(d_u dv) \leq \pi^v_{ij} \leq c_2/(d_u d_v)$, then

$$\frac{1}{d_u D}\|\widehat{\mathcal{M}} - \mathcal{M}\|_F^2 \lesssim \frac{\text{rank}(\mathcal{M})}{p(d_u \wedge D)}.$$

- We denote $n = \sum_{v\in[V]} \sum_{(i,j)\in[d_u]\times[d_v]} \pi^v_{ij}$, the expected number of observations.
- **Sample complexity:**

$$n \gtrsim \text{rank}(\mathcal{M})(d_u \vee D).$$

Example: 1-bit matrix completion

- **1-bit matrix completion:** $\mathcal{Y} \in \{+1, -1\}$ with probability $f(\mathcal{M})$ for some link-function f [Davenport et al. (2014); Klopp et al. (2015); Alquier et al. (2017)]
- Klopp et al. (2015) obtained the rate $\mathrm{rank}(\mathcal{M})(d \vee D) \log(d \vee D)/n$ as the upper bound and $\mathrm{rank}(M)(d \vee D)/n$ as the lower bound for 1-bit matrix completion.

Corollary[A., Klopp 2018]

$$\frac{1}{dD}\|\widehat{\mathcal{M}} - \mathcal{M}\|_F^2 \lesssim \frac{\mathrm{rank}(\mathcal{M})(d \vee D)}{n},$$

- **Answer** the important theoretical question: what is the exact minimax rate of convergence for 1-bit matrix completion which was previously known up to a logarithmic factor.
- **Sum-norm penalization:** $\sum_{v \in [V]} \|\boldsymbol{M}^v\|_*$

Distribution-free setting

- We do not assume any specific model for the observations.
- We consider the risk of estimating $\boldsymbol{X}^v$ with a loss function ℓ^v,
- We focus on non-negative loss functions that are Lipschitz:

Assumption 3: We assume that the loss function $\ell^v(y, \cdot)$ is ρ_v-Lipschitz in its second argument: $\ell^v(y, x) - \ell^v(y, x')| \leq \rho_v |x - x'|$.

- Examples: hinge loss with $\ell^v(y, y') = \max(0, 1 - yy')$, logistic loss with $\ell^v(y, y') = \log(1 + \exp(-yy'))$, etc.

Distribution-free setting: estimation procedure

- Goodness-of-fit term:

$$R_{\mathcal{Y}}(\mathcal{W}) = \frac{1}{d_u D} \sum_{v \in [V]} \sum_{(i,j) \in [d_u] \times [d_v]} B_{ij}^v \ell^v(Y_{ij}^v, W_{ij}^v).$$

- We define the oracle as:

$$\overset{\star}{\mathcal{M}} = (\overset{\star}{M}{}^1, \ldots, \overset{\star}{M}{}^V) = \operatorname*{argmin}_{\|\mathcal{W}\|_\infty \le \gamma} R(\mathcal{W}),$$

where $R(\mathcal{W}) = \mathbb{E}[R_{\mathcal{Y}}(\mathcal{W})]$.

- For a tuning parameter $\Lambda > 0$, the nuclear norm penalized estimator $\widehat{\mathcal{M}}$ is defined as

$$\widehat{\mathcal{M}} \in \operatorname*{argmin}_{\|\mathcal{W}\|_\infty \le \gamma} \big\{ R_{\mathcal{Y}}(\mathcal{W}) + \Lambda \|\mathcal{W}\|_* \big\}.$$

Distribution-free setting: theoretical guarantee

- We denote by $\|\mathcal{W}\|^2_{\Pi,F} = \sum_v \sum_{(i,j)} \pi^v_{ij}(W^v_{ij})^2$.

Assumption 4: Assume that for every $\mathcal{W}$ with $\|\mathcal{W}\|_\infty \leq \gamma$, one has $R(\mathcal{W}) - R(\overset{\star}{\mathcal{M}}) \gtrsim \frac{1}{d_u D}\|\mathcal{W} - \overset{\star}{\mathcal{M}}\|^2_{\Pi,F}$.

- Assumption 4 is called "Bernstein" condition (Mendelson, 2008; Bartlett et al., 2004; Alquier et al., 2017; Elsener and van de Geer, 2018).

Theorem [A. Klopp 2018]

Let Assumptions 1, 3, and 4 hold and $\Lambda \approx (\sqrt{\mu} + \sqrt{\log(d_u \vee D)})/(d_u D)$. Then, with probability , one has

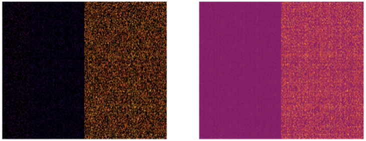
$$R(\widehat{\mathcal{M}}) - R(\overset{\star}{\mathcal{M}}) \lesssim \frac{\mu + \log(d_u \vee D)}{p d_u D}$$

Toy example using CVXPY package

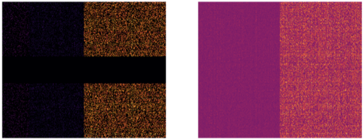
V	d_u	d_1	d_2	d_3	$M^1(\text{rank} = 5)$	$M^2(\text{rank} = 10)$	$M^3(\text{rank} = 15)$
3	500	100	200	300	$\mathcal{N}(-2, 0.5)$	$\mathcal{N}(1, 0.5)$	$\mathcal{N}(2, 0.5)$

	M^1	M^2	M^3	$\mathcal{M}$	$\mathcal{M}_{\text{cold}}$
% observations	10%	20%	30%	23.29%	18.69%

	CMC	SNN	Cold-Start	$\widehat{M^1}$	$\widehat{M^2}$	$\widehat{M^3}$
RMSE	0.223	0.224	0.220	0.198	0.194	0.311



observed + fitted collective matrix.



observed + fitted cold collective matrix.

CVXPY [Diamond and S. Boyd (2016)]

Take Home Message

- First theoretical guarantees on the case of noisy collective MC.
- Collective approach provides faster rate of convergences in the case of joint low-rank structure.
- Exact minimax optimal rate of convergence for 1-bit matrix completion which was known upto a logarithmic factor.
- On going work: algorithmic study with numerical experiments.

Thank you.

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