Binarsity: A Penalization for One-Hot Encoded Features in Linear Supervised Learning

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Who I am?

- Mokhtar Z. Alaya
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Join work with

- Dr. Simon Bussy
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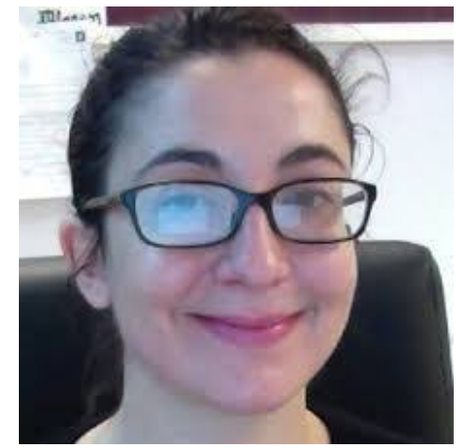


- Pr. Agathe Guilloux
- Professor at LaMME
- Univ. Evry Val d'Essonne

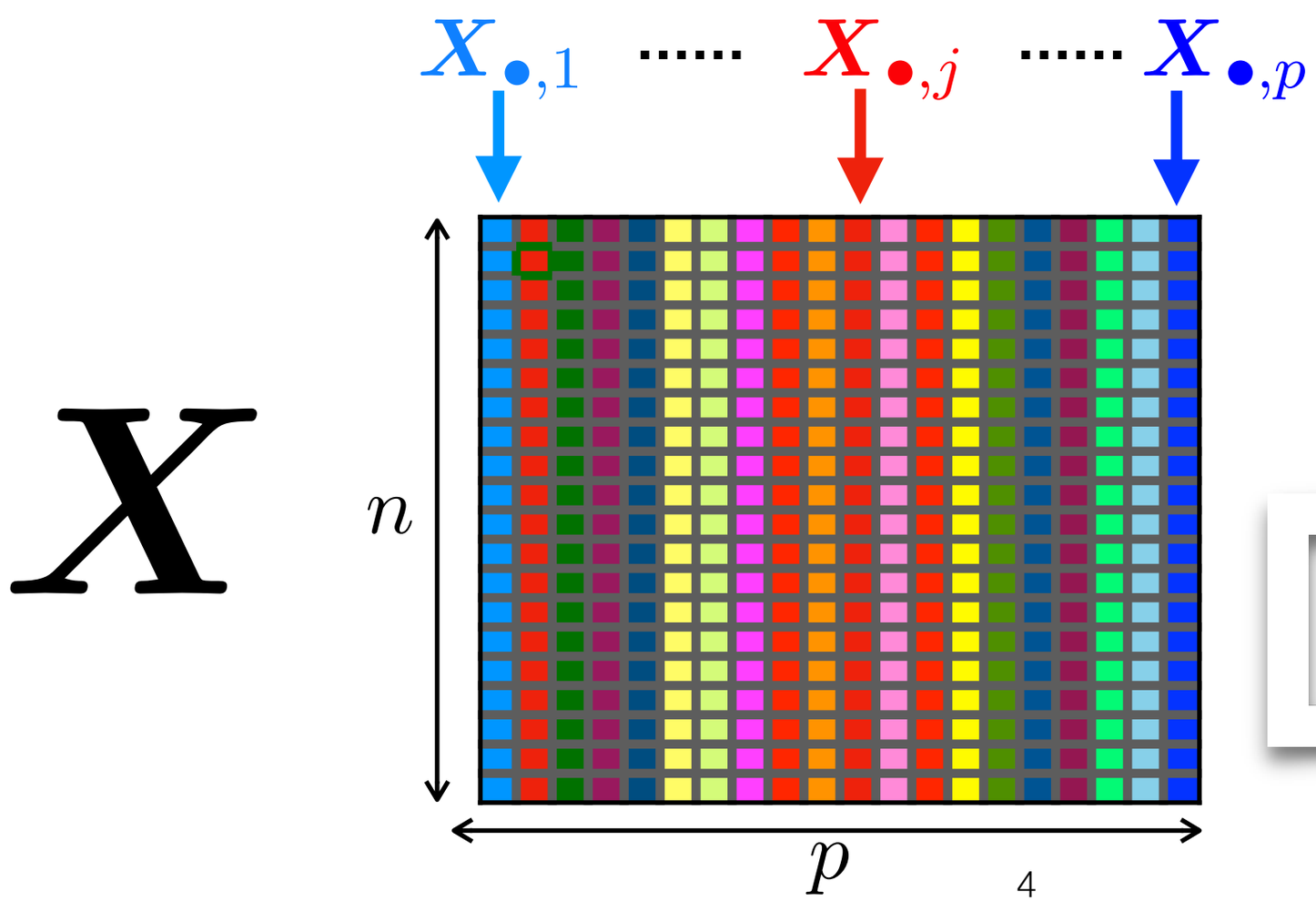
Supervised Learning: Setting

Supervised training dataset
$\mathcal{D}_n = \{(x_i, y_i) : i = 1, \ldots, n\}$

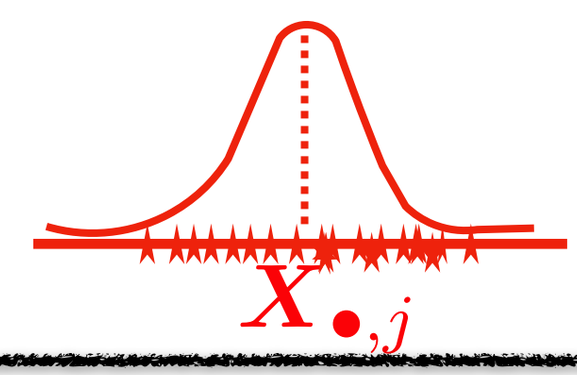
with features
$x_i = [x_{i,1}, \ldots, x_{i,p}]^\top \in \mathbb{R}^p$

and labels
$y_i \in \mathcal{Y} \subset \mathbb{R}$

Features matrix
$$\boldsymbol{X} = [x_{i,j}]_{1 \le i \le n; 1 \le j \le p} \in \mathbb{R}^{n \times p}$$



Several continuous features



A well know-trick:
One-Hot Encoding

(Lieu et al., '02);

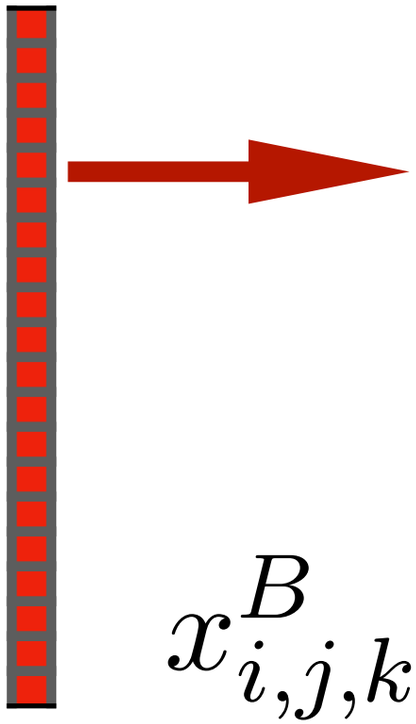
Wu and Coggeshall, '12)

One-Hot Encoding: Features Binarization

Features Binarization: Setup

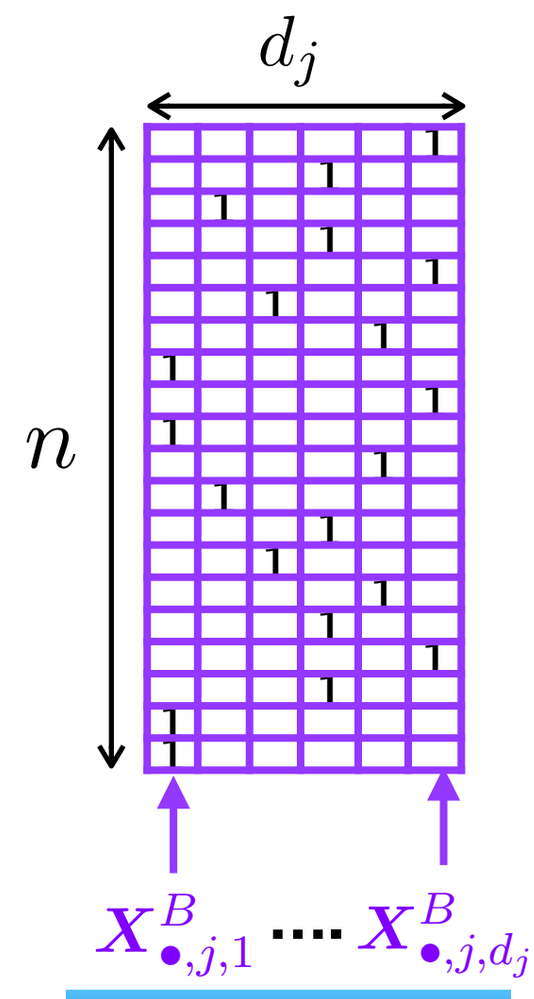
Binarization Setup:
The j-th column $\boldsymbol{X}_{\bullet,j}$ is replaced by a number $d_j \geq 2$ of binary columns (containing only zeros and ones)
$\boldsymbol{X}^B_{\bullet,j,1}, \ldots, \boldsymbol{X}^B_{\bullet,j,d_j}$

$\boldsymbol{X}_{\bullet,j}$



Partition of the range of values of $\boldsymbol{X}_{\bullet,j}$ into intervals $I_{j,1}, \ldots, I_{j,d_j}$ and put

$$x^B_{i,j,k} = \begin{cases} 1, & \text{if } x_{i,j} \in I_{j,k}, \\ 0, & \text{otherwise} \end{cases}$$

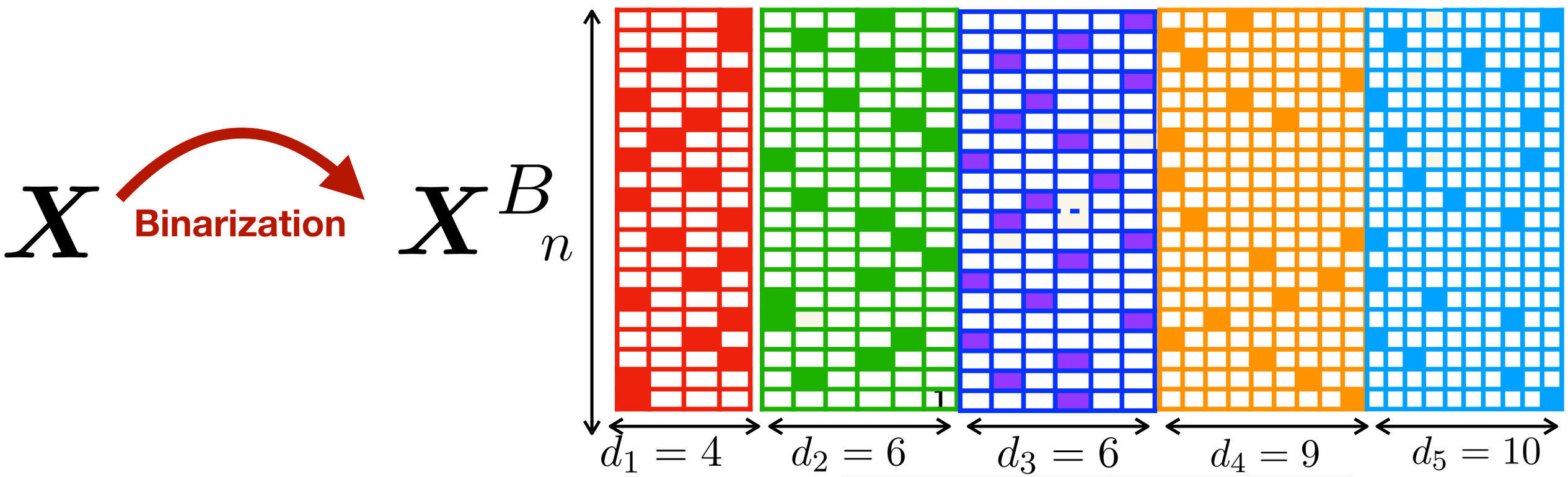


Binary Features

Features Binarization: Setup

If $\boldsymbol{X}_{\bullet,j}$ takes values **(modalities)** in the set $\{1, \ldots, M_j\}$ with cardinality M_j, we take $d_j = M_j$ and use one-hot coding of each modality by defining:

$$x^B_{i,j,k} = \begin{cases} 1, & \text{if } x_{i,j} = k, \\ 0, & \text{otherwise} \end{cases}$$



Binarized Features Matrix

Features Binarization: Inter-quantile Partition Intervals

The i-th of the binarized matrix $\boldsymbol{X}^B$ reads as

$$x_i^B = [x_{i,1,1}^B, \ldots, x_{i,1,d_1}^B, x_{i,2,1}^B, \ldots, x_{i,2,d_2}^B, \ldots, x_{i,p,1}^B, \ldots, x_{i,p,d_p}^B]^\top \in \mathbb{R}^d$$

1 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0

where

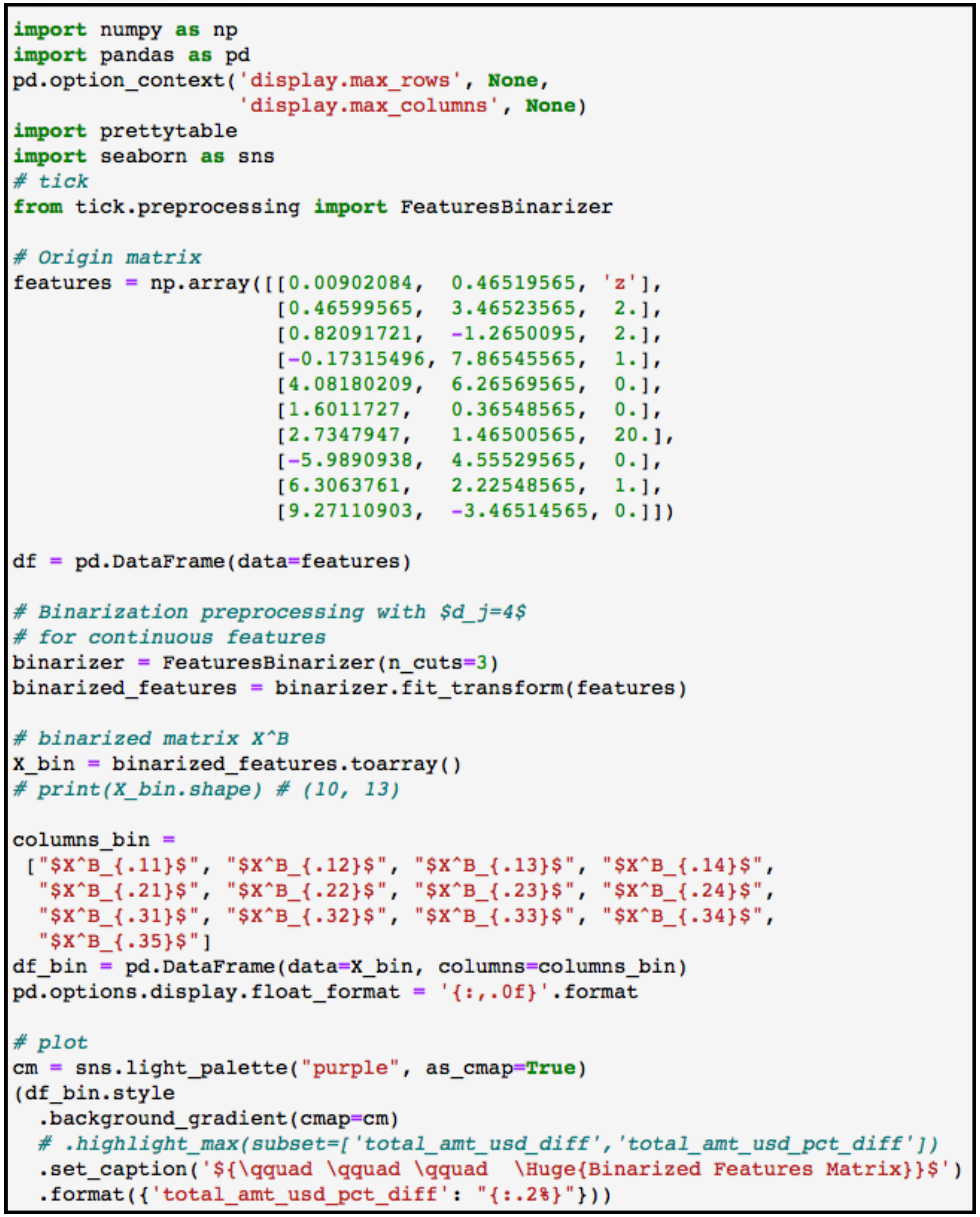
$$\boxed{d = \sum_{j=1}^{p} d_j}$$

Choice of the $I_{j,k}$ Intervals? → **Natural Choice: Inter-quantile intervals**

$$I_{j,1} = \left[q_j(0), q_j(\frac{1}{d_j})\right] \text{ and } I_{j,k} = \left(q_j(\frac{k-1}{d_j}), q_j(\frac{k}{d_j})\right]$$

for $k = 2, \ldots, d_j$, and where $q_j(\alpha)$ denotes a quantile of order $\alpha \in [0,1]$ for $\boldsymbol{X}_{\bullet,j}$

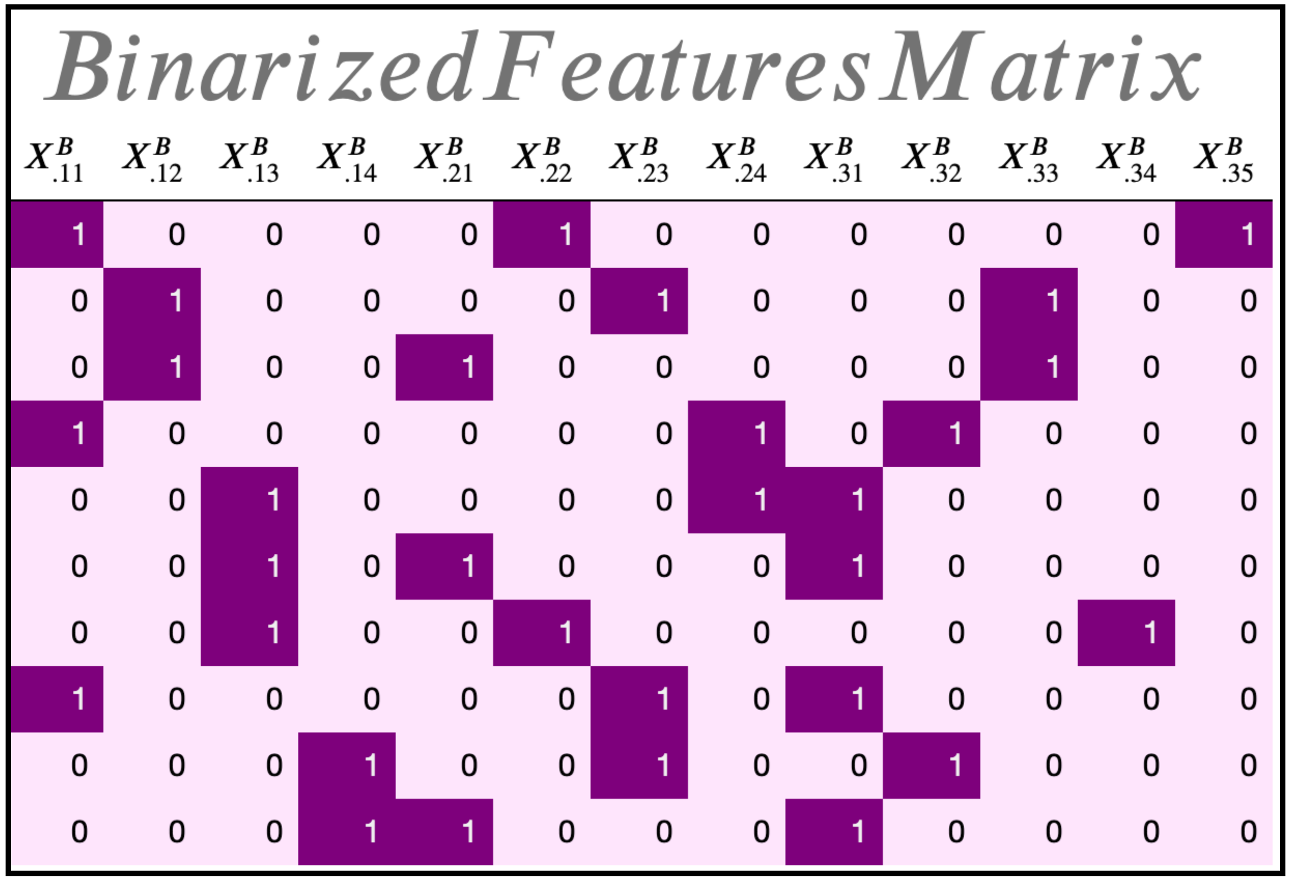
Features Binarization: Example



```
import numpy as np
import pandas as pd
pd.option_context('display.max_rows', None,
                  'display.max_columns', None)
import prettytable
import seaborn as sns
# tick
from tick.preprocessing import FeaturesBinarizer

# Origin matrix
features = np.array([[0.00902084,  0.46519565, 'z'],
                     [0.46599565,  3.46523565,  2.],
                     [0.82091721,  -1.2650095,  2.],
                     [-0.17315496, 7.86545565,  1.],
                     [4.08180209,  6.26569565,  0.],
                     [1.6011727,   0.36548565,  0.],
                     [2.7347947,   1.46500565,  20.],
                     [-5.9890938,  4.55529565,  0.],
                     [6.3063761,   2.22548565,  1.],
                     [9.27110903,  -3.46514565, 0.]])

df = pd.DataFrame(data=features)

# Binarization preprocessing with $d_j=4$
# for continuous features
binarizer = FeaturesBinarizer(n_cuts=3)
binarized_features = binarizer.fit_transform(features)

# binarized matrix X^B
X_bin = binarized_features.toarray()
# print(X_bin.shape) # (10, 13)

columns_bin =
 ["$X^B_{.11}$", "$X^B_{.12}$", "$X^B_{.13}$", "$X^B_{.14}$",
  "$X^B_{.21}$", "$X^B_{.22}$", "$X^B_{.23}$", "$X^B_{.24}$",
  "$X^B_{.31}$", "$X^B_{.32}$", "$X^B_{.33}$", "$X^B_{.34}$",
  "$X^B_{.35}$"]
df_bin = pd.DataFrame(data=X_bin, columns=columns_bin)
pd.options.display.float_format = '{:,.0f}'.format

# plot
cm = sns.light_palette("purple", as_cmap=True)
(df_bin.style
  .background_gradient(cmap=cm)
  # .highlight_max(subset=['total_amt_usd_diff','total_amt_usd_pct_diff'])
  .set_caption('${\qquad \qquad \qquad  \Huge{Binarized Features Matrix}}$')
  .format({'total_amt_usd_pct_diff': "{:.2%}"}))
```

tick
ML Python Package

(Bacry et al., 2018)



Binarized Features Matrix

$X^B_{.11}$	$X^B_{.12}$	$X^B_{.13}$	$X^B_{.14}$	$X^B_{.21}$	$X^B_{.22}$	$X^B_{.23}$	$X^B_{.24}$	$X^B_{.31}$	$X^B_{.32}$	$X^B_{.33}$	$X^B_{.34}$	$X^B_{.35}$
1	0	0	0	0	1	0	0	0	0	0	0	1
0	1	0	0	0	0	1	0	0	0	1	0	0
0	1	0	0	1	0	0	0	0	0	1	0	0
1	0	0	0	0	0	0	1	0	1	0	0	0
0	0	1	0	0	0	0	1	1	0	0	0	0
0	0	1	0	1	0	0	0	1	0	0	0	0
0	0	1	0	0	1	0	0	0	0	0	1	0
1	0	0	0	0	0	1	0	1	0	0	0	0
0	0	0	1	0	0	1	0	0	1	0	0	0
0	0	0	1	1	0	0	0	1	0	0	0	0

Features Binarization: Weights

Weights of one-hot encoded features

To each binarized feature $\boldsymbol{X}^B_{\bullet,j,k}$ corresponds a parameter $\theta_{j,k}$

The parameters associated to the binarization of the j-th feature is denoted

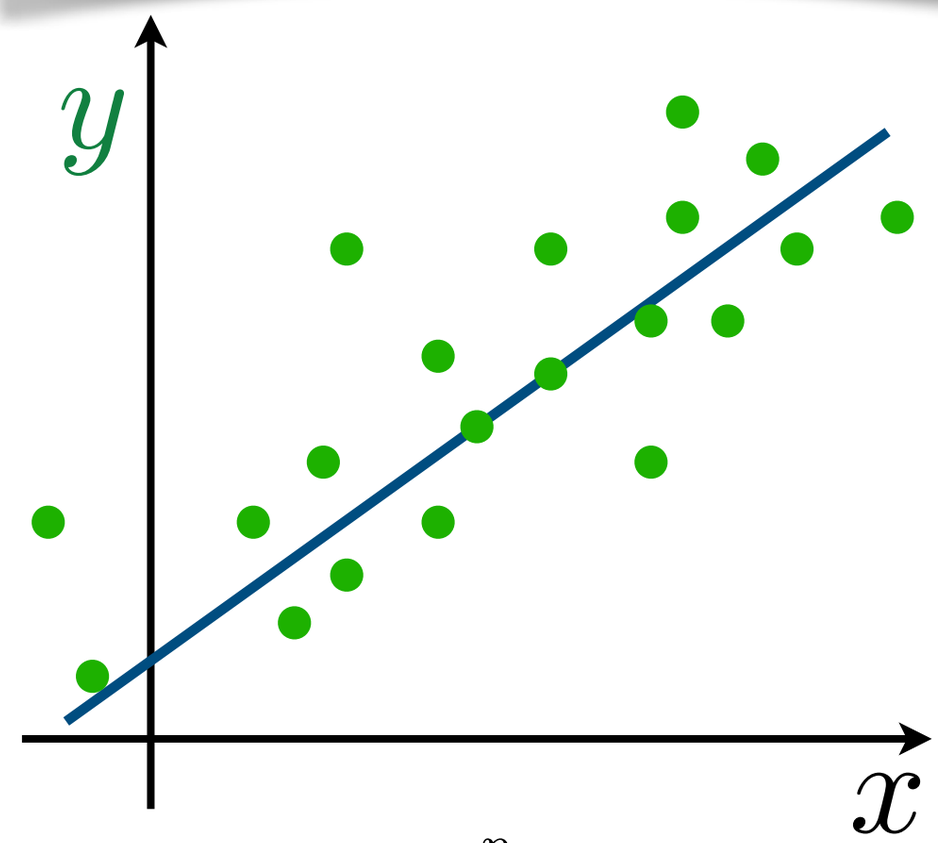
$$\theta_{j,\bullet} = [\theta_{j,1} \ \cdots \ \theta_{j,d_j}]^\top$$

The full parameters vector of size $d = \sum_{j=1}^{p} d_j$, is simply

$$\theta = [\theta_{1,1} \ \cdots \ \theta_{1,d_1} \ \theta_{2,1} \ \cdots \ \theta_{2,d_2} \ \cdots \ \theta_{p,1} \ \cdots \ \theta_{p,d_p}]^\top \in \mathbb{R}^d$$

Features Binarization: Weights

Linear regression on raw features

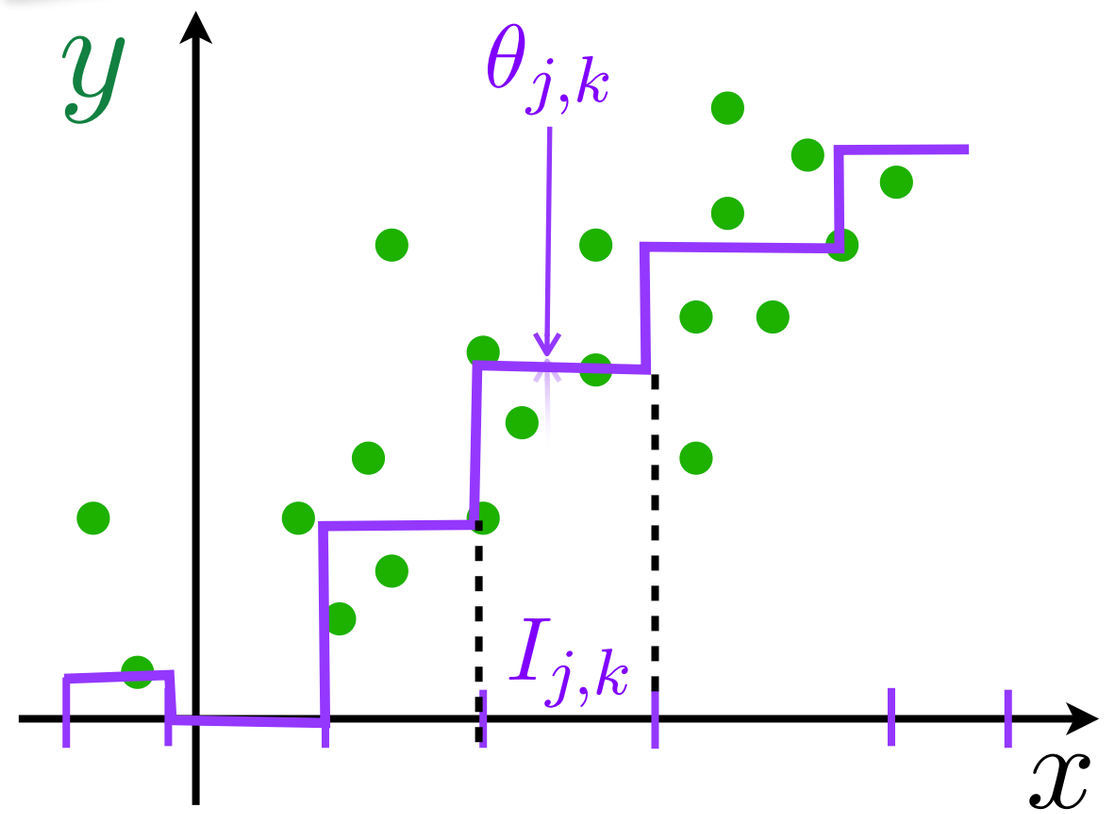


$$y_i = \omega^\top x_i + b = \sum_{j=1}^{p} \omega_j x_{i,j} + b$$

*Impact of the j-th feature is **linear** and encoded by a **single weight** ω_j*

$$x \mapsto \omega_j x$$

Linear regression on binarized features



$$y_i = \sum_{j=1}^{p} \sum_{k=1}^{d_j} \theta_{j,k} x_{i,j,k} + b = \sum_{j=1}^{p} \sum_{k=1}^{d_j} \theta_{j,k} \mathbb{1}(x_{i,j} \in I_{j,k}) + b$$

*Impact of the j-th feature is **piecewise constant** and encoded by a **block***

$$\theta_{j,\bullet} = [\theta_{j,1} \cdots \theta_{j,d_j}]^\top$$

$$x \mapsto \sum_{k=1}^{d_j} \theta_{j,k} \mathbb{1}(x \in I_{j,k})$$

Features Binarization: Issues

(P1) Colinear binary features
One-hot-encodings satisfy

$$\sum_{k=1}^{d_j} x^B_{i,j,k} = 1, \text{ for all } j = 1, \ldots, p$$

$\rightarrow X^B$ **Not full rank**

(P2) Overparametrization
Increasing the number of bins d_j

$\rightarrow$ **Overfitting**

(P3) Feature selection
Some of the raw features $X_{\bullet,j}$ might be not relevant for the prediction task!

$\rightarrow$ **Block-Sparsity!**

$$\theta_{j,1} = 0, \ldots, \theta_{j,d_j} = 0$$

Features Binarization: Solutions

To deal with **(P1)**, we impose a **linear constraint** in each block (Agresti, 2015)

(S1) $$n_j^\top \theta_{j,\bullet} = \sum_{k=1}^{d_j} n_{j,k}\theta_{j,k} = 0 \text{ for all } j = 1, \ldots, p$$

$$n_j = [n_{j,1}, \ldots, n_{j,d_j}]^\top \in \mathbb{N}^{d_j} \quad \text{where} \quad n_{j,k} = |\{i : x_{i,j} \in I_{j,k}\}|$$

To tackle **(P2)**, we keep the number of different values taken by $\theta_{j,\bullet}$ to minimal level by using a within **block weighted total-variation penalization**

(S2) $$\sum_{j=1}^{p} \|\theta_{j,\bullet}\|_{\mathrm{TV},\hat{\omega}_{j,\bullet}} = \sum_{k=2}^{d_j} \hat{\omega}_{j,k}|\theta_{j,k} - \theta_{j,k-1}|$$

(S1) + (S2) solve **(P3)**

Binarsity

$$\mathrm{bina}(\theta) = \sum_{j=1}^{p} \left(\sum_{k=2}^{d_j} \hat{\omega}_{j,k} |\theta_{j,k} - \theta_{j,k-1}| + \delta_j(\theta_{j,\bullet}) \right)$$

where

$$\delta_j(u) = \begin{cases} 0 & \text{if} \quad n_j^\top u = 0, \\ \infty & \text{otherwise} \end{cases}$$

and $\hat{\omega}_{j,k} = \mathcal{O}\left(\sqrt{\frac{\log d}{n} \hat{\pi}_{j,k}} \right)$ with $\hat{\pi}_{j,k} = \frac{\left| \left\{ i = 1, \ldots, n : x_{i,j} \in \cup_{k'=k}^{d_j} I_{j,k'} \right\} \right|}{n}$

$\hat{\pi}_{j,k}$ = Proportion of **1**'s in the sub-matrix obtained by deleting the first k columns in the j-th binarized block matrix

$X^B_{.11}$	$X^B_{.12}$	$X^B_{.13}$	$X^B_{.14}$	$X^B_{.21}$	$X^B_{.22}$	$X^B_{.23}$	$X^B_{.24}$	$X^B_{.31}$	$X^B_{.32}$	$X^B_{.33}$	$X^B_{.34}$	$X^B_{.35}$
1	0	0	0	0	1	0	0	0	0	0	0	1
0	1	0	0	0	0	1	0	0	0	1	0	0
0	1	0	0	1	0	0	0	0	0	1	0	0
1	0	0	0	0	0	0	1	0	1	0	0	0
0	0	1	0	0	0	0	1	1	0	0	0	0
0	0	1	0	1	0	0	0	1	0	0	0	0
0	0	1	0	0	1	0	0	0	0	0	1	0
1	0	0	0	0	0	1	0	1	0	0	0	0
0	0	0	1	0	0	1	0	0	1	0	0	0
0	0	0	1	1	0	0	0	1	0	0	0	0

Binarsity: Interpretation

Large d_j → $\theta_{j,k}$ and $\theta_{j,k-1}$ to be close

Penalization ℓ_1 → **Sparsity**

Penalization $\boldsymbol{Binarsity}$ → **Piecewise constant** $\theta_{j,\bullet}$

If $\theta_{j,\bullet}$ is constant than the linear constraint ${n_j}^\top \theta_{j,\bullet} = 0$ entails $\theta_{j,\bullet} \equiv 0$

→ **Block-Sparsity!**

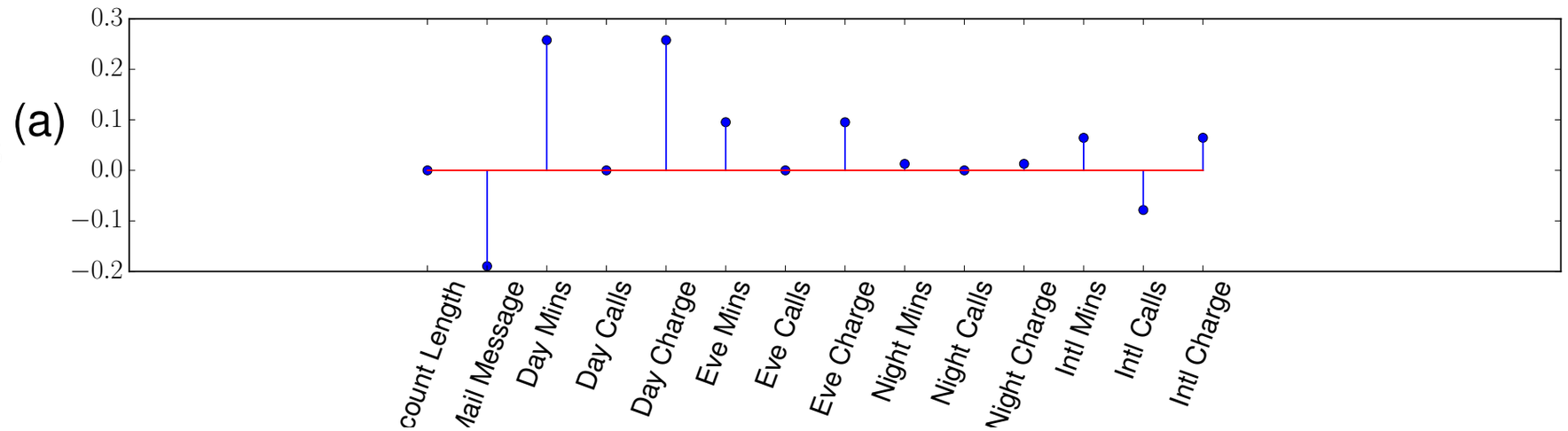
$\boldsymbol{X}_{\bullet,j}$ → Statistically relevant if $\theta_{j,\bullet} \not\equiv 0$

$\boldsymbol{X}_{\bullet,j}$ → Statistically not relevant if $\theta_{j,\bullet} \equiv 0$

Illustrations

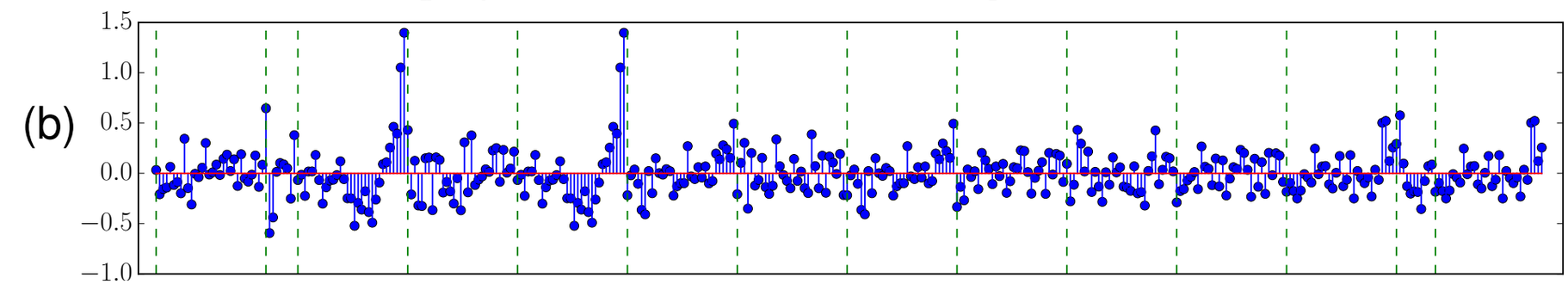
Weights of a logistic Regression on Churn Dataset (UCI) $n = 3333, p = 14$

Raw continuous features



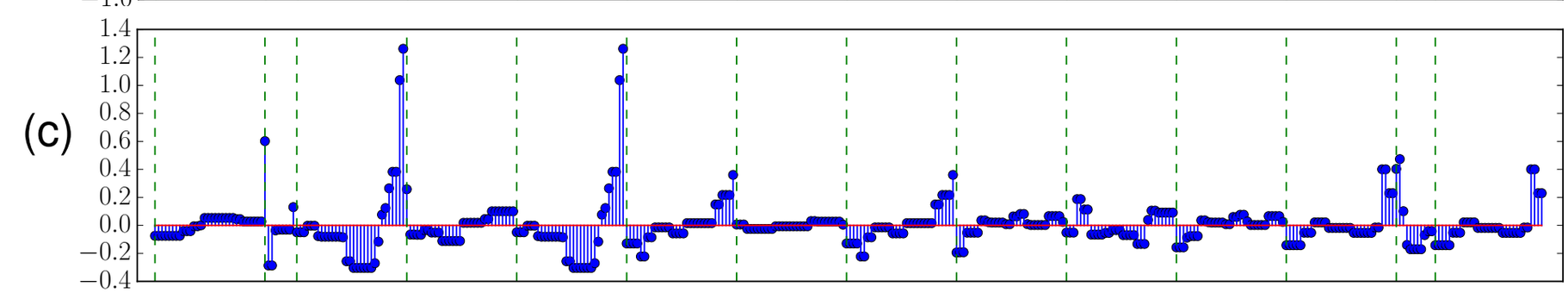
Binarized features

No-penalization



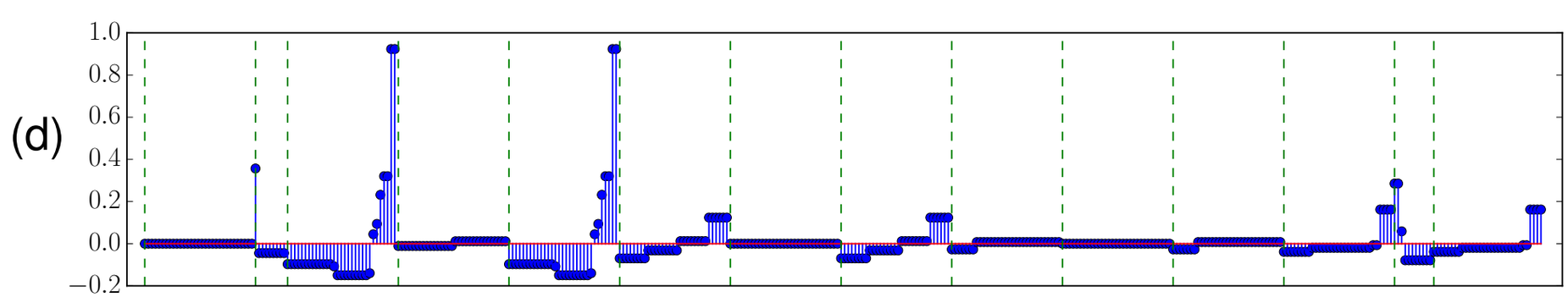
Binarized features

Low binarsity penalization



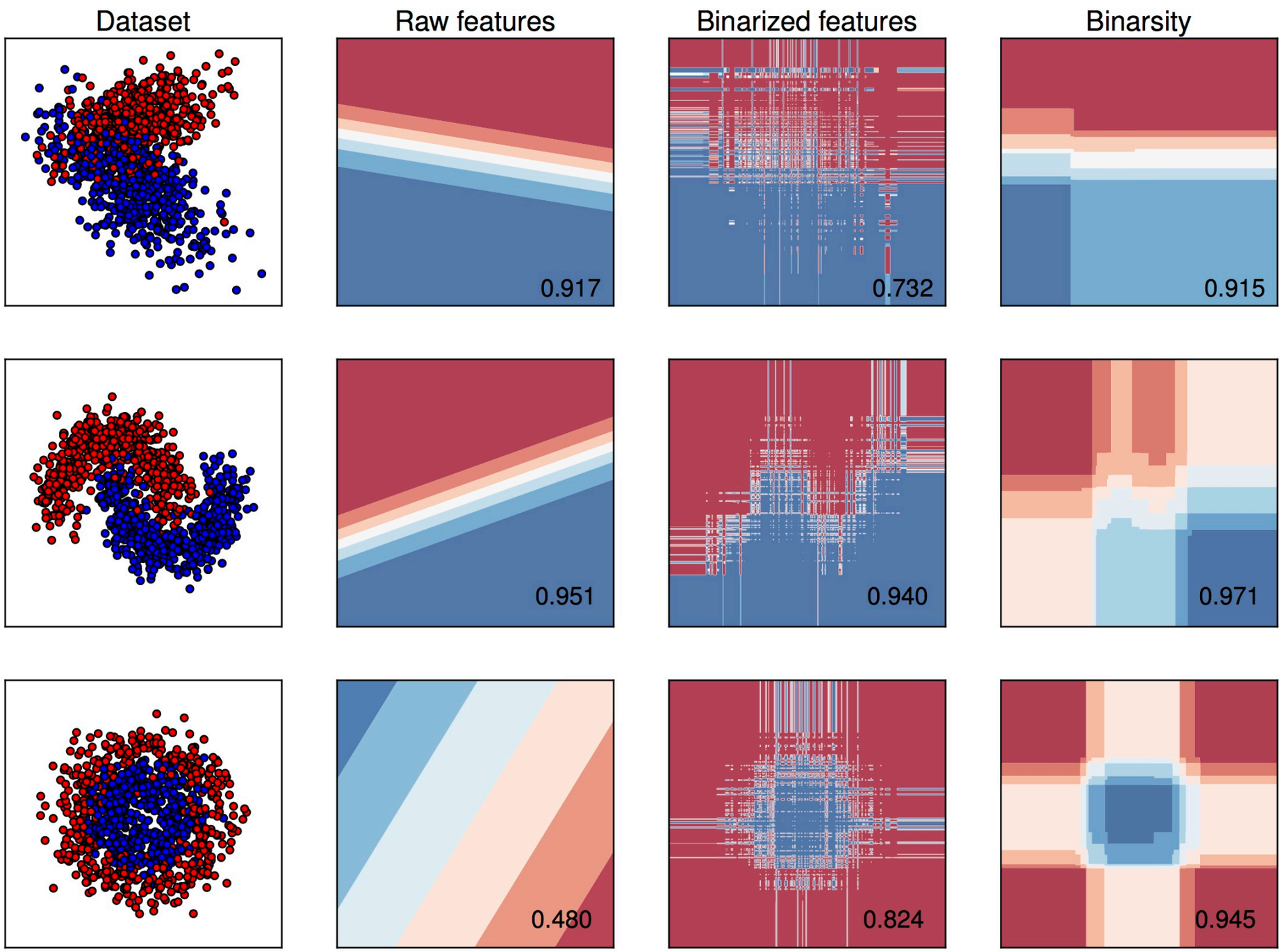
Binarized features

Strong binarsity penalization



Decision Boundaries of a Logistic Regression

Toy datasets with $n = 1000, p = 2$ and $d_1 = d_2 = 100$



Theoretical Guarantees (GLM + Binarsity)

Generalized Linear Models

$$\mathbb{P}(y|x) = \exp\left(\frac{y m^0(x) - b(m^0(x))}{\phi} + c(y, \phi)\right)$$

The functions $b(\cdot)$ and $c(\cdot)$ and the dispersion parameter ϕ are **known**.

The natural parameter $m^0(\cdot)$ is **unknown** with

$$m^0(x) = g(\mathbb{E}[y|x]), \text{ where } b' = g^{-1}$$

Examples:
Logistic and probit regression for binary data or multinomial regression for categorical data, Poisson regression for count data, etc ...

GLM: Goodness-of-fit

Empirical risk

$$R_n(m_\theta) = \frac{1}{n}\sum_{i=1}^{n} \ell(y_i, m_\theta(x_i))$$

$$m_\theta(x) = \theta^\top x^B$$

GLM loss function $\ell(y, y') = -yy' + b(y')$

We estimate m^0 by $\hat{m} = m_{\hat{\theta}}$

$$\hat{\theta} \in \operatorname*{argmin}_{\theta \in \mathbb{R}^d} \left\{ R_n(m_\theta) + \mathrm{bina}(\theta) \right\}$$

Assumptions on GLM

A1. $b(\cdot)$ is three times continuously

A2. $|b'''(z)| \leq C_b |b''(z)|$ for some $C_b > 0$

A3. $C_n = \max_{i=1,\ldots,n} |m^0(x_i)| < \infty$

A4. $L_n \leq \max_{i=1,\ldots,n} b''\big(m^0(x_i)\big) \leq U_n$

- Satisfied for the following standard GLM :

Model	ϕ	$b(z)$	$b'(z)$	$b''(z)$	$b'''(z)$	C_b	L_n	U_n
Normal	σ^2	$\frac{z^2}{2}$	z	1	0	0	1	1
Logistic	1	$\log(1+e^z)$	$\frac{e^z}{1+e^z}$	$\frac{e^z}{(1+e^z)^2}$	$\frac{1-e^z}{1+e^z} b''(z)$	2	$\frac{e^{C_n}}{(1+e^{C_n})^2}$	1/4
Poisson	1	e^z	e^z	e^z	e^z	1	e^{-C_n}	e^{C_n}

Orcale inequality

Non asymptotic oracle inequality in terms of **Excess risk**:

$$R(\hat{m}) - R(m_0) = \mathbb{E}_{\mathbb{P}(y|x)}[R_n(\hat{m}) - R_n(m_0)]$$

Non asymptotic oracle inequality

- How close is $\hat{m}$ to the expected minimal risk?
- Measured by the excess risk $R(\hat{m}) - R(m_0)$
  - For n fixed prove some thing like

$$R(\hat{m}) - R(m_0) \leq \inf_{\theta} \left\{ R(\hat{m}) - R(m_0) + \frac{\text{complexity}(\theta)}{n} \right\}$$

  - Lasso (Bickel, Ritov, Tsybakov '09): $\text{complexity}(\theta) = s(\theta) \log p$

$$\text{where } s(\theta) = \text{sparsity}(\theta) = |\{j = 1, \ldots, p : \theta_j \neq 0\}$$

Binarsity: new measure of sparsity

For $\theta \in \mathbb{R}^d$, let $J(\theta) = \big[J_1(\theta), \ldots, J_p(\theta)\big]$ be the concatenation of the support sets relative to the total-variation penalization, that is

$$J_j(\theta) = \big\{k : \theta_{j,k} \neq \theta_{j,k-1}, \text{ for } k = 2, \ldots, d_j\big\}.$$

$$\boxed{\boldsymbol{binarsity}(\theta) = |J(\theta)| = \sum_{j=1}^{p} |J_j(\theta)| = \sum_{j=1}^{p} \big|\big\{k : \theta_{j,k} \neq \theta_{j,k-1}, \text{ for } k = 2, \ldots, d_j\big\}\big|}$$

counts the number of non-equal consecutive values of θ

- If θ is block-sparse $|\mathcal{J}(\theta)| \ll p$ where $|\mathcal{J}(\theta) = \{j = 1, \ldots, p : \theta_{j,\bullet} \neq \mathbf{0}_{d_j}\}$

then $$|J(\theta)| \leq |\mathcal{J}(\theta)| \max_{j \in \mathcal{J}(\theta)} |J_j(\theta)|$$

which means that $|J(\theta)|$ is controlled by the block-sparsity $|\mathcal{J}(\theta)|$

Restricted Eigenvalues Assumption

- Let $K = [K_1, \ldots, K_p]$ be a concatenation of index sets and define

$$\kappa(K) \in \inf_{u \in \mathscr{C}_{\mathrm{TV},\hat{\omega}}(K) \setminus \{\mathbf{0}_d\}} \left\{ \frac{\|\boldsymbol{X}^B u\|_2}{\sqrt{n}\|u_K\|_2} \right\}$$

with

$$\mathscr{C}_{\mathrm{TV},\hat{\omega}}(K) = \left\{ u \in \mathbb{R}^d : \sum_{j=1}^p \|(u_{j,\bullet})_{K_j^{\mathsf{c}}}\|_{\mathrm{TV},\hat{\omega}_{j,\bullet}} \leq 2 \sum_{j=1}^p \|(u_{j,\bullet})_{K_j}\|_{\mathrm{TV},\hat{\omega}_{j,\bullet}} \right\}.$$

where $(u_K)_k = u_k$ if $k \in K$ and $(u_K)_k = 0$ if $k \notin K$.

- The $\mathscr{C}_{\mathrm{TV},\hat{\omega}}(K)$ is a cone composed by all vectors with a support "close" to K.

Assumption: We have $\kappa(K) > 0$ for any K such that $|K| = \sum_{j=1}^p |K_j| \leq J^\star$.

Theorem

- Define

$$\hat{\omega}_{j,k} = \sqrt{\frac{2U_n\phi(A + \log d)}{n}}\hat{\pi}_{j,k}$$

for some constant $A > 0$ and consider

$$\hat{\theta} \in \underset{\theta \in B_d(\rho)}{\operatorname{argmin}} \left\{ R_n(m_\theta) + \operatorname{bina}(\theta) \right\}$$

where

$$B_d(\rho) = \{\theta \in \mathbb{R}^d : \textstyle\sum_{j=1}^p \|\theta_{j,\bullet}\|_\infty \leq \rho\}.$$

- A standard constraint in literature for the proof of oracle inequalities for sparse GLMs (Van de Geer '08), (Ivanoff et al. '16).

- Note that

$$\max_{i=1,\ldots,n} |\langle x_i^B, \theta\rangle \leq \textstyle\sum_{j=1}^p \|\theta_{j,\bullet}\|_\infty \leq |\mathcal{J}(\theta)| \times \|\theta\|_\infty.$$

  - So $\theta \in B_d(\rho)$ is entailed by a box constraint on θ, which depends on the dimensionality of the features through $|\mathcal{J}(\theta)|$.

Theorem

$$R(\hat{m}) - R(m^0) \leq \inf_{\substack{\theta \in B_d(\rho) \\ \forall j \ \mathbf{1}^\top \theta_{j,\bullet} = 0 \\ |J(\theta)| \leq J^\star}} \Big\{ 3(R(m_\theta) - R(m^0)) + \frac{\xi |J(\theta)|}{\kappa^2(J(\theta))} \max_{j=1,\ldots,p} \|(\hat{\omega}_{j,\bullet})_{J_j(\theta)}\|_\infty^2 \Big\},$$

with probability at least $1 - 2e^{-A}$ and $\xi = cst(C_n, \rho, L_n, U_n)$.

- The complexity term satisfies

$$|J(\theta)| \max_{j=1,\ldots,p} \|(\hat{\omega}_{j,\bullet})_{J_j(\theta)}\|_\infty^2 \leq 2U_n \phi \frac{|J(\theta)|(A + \log d)}{n}.$$

- $\hat{\theta}$ achieves an optimal tradeoff between bias $R(m_\theta) - R(m^0)$ and sparsity $|J(\theta)|$.

- Fast Rate $\approx \big(\boldsymbol{binarsity}(\theta) \times \log(d)\big)/n$

Implementations

Optimization problem

Need to optimize

$$R_n(\theta) + \mathrm{bina}(\theta)$$

Smooth (Gradient Lipschitz)

Non-differentiable

First order optimization: Proximal Gradient Descent (Bach et al., 2012)

$$\theta^{(t+1)} \leftarrow \mathrm{prox}_{\eta\,\mathrm{bina}(\cdot)}\left(\theta^{(t)} - \eta \nabla R_n(\theta^{(t)})\right)$$

proximal operator

learning rate

$$\mathrm{prox}_{\lambda g}(y) = \underset{\theta \in \mathbb{R}^n}{\mathrm{argmin}} \left\{ \frac{1}{2}\|y - \theta\|_2^2 + \lambda g(\theta) \right\}, \text{ for all } y \in \mathbb{R}^n.$$

Proximal Operator of Binarsity

Binarsity is separable by blocks, then for all j=1, …, p,

$$\left(\operatorname{prox}_{\text{bina}}(\theta)\right)_{j,\bullet} = \operatorname{prox}_{(\|\cdot\|_{\text{TV},\hat{\omega}_{j,\bullet}}+\delta_j)}(\theta_{j,\bullet})$$

The following algorithm expresses $\operatorname{prox}_{\text{bina}}$

Algorithm 1:

Input: vector $\theta \in \mathbb{R}^d$, weights $\hat{\omega}_{j,k}$ and $n_{j,k}$ for $j = 1, \ldots, p$ and $k = 1, \ldots, d_j$

Output: vector $\eta = \operatorname{prox}_{\text{bina}}(\theta)$

for $j = 1$ ***to*** p **do**

- $\beta_{j,\bullet} \leftarrow \operatorname{prox}_{\|\theta_{j,\bullet}\|_{\text{TV},\hat{\omega}_{j,\bullet}}}(\theta_{j,\bullet})$ (weighted TV penalization in block $\theta_{j,\bullet}$)
- $\eta_{j,\bullet} \leftarrow \beta_{j,\bullet} - \frac{n_j \top \beta_{j,\bullet}}{\|n_j\|_2^2} n_j$ (project onto $\operatorname{span}(n_j)^{\perp}$)

Return: η

Proximal Operator of Weighted TV

$$\hat{\theta} = \mathrm{prox}_{\|\cdot\|_{\mathrm{TV},\hat{\omega}}}(y) = \underset{\theta \in \mathbb{R}^n}{\mathrm{argmin}} \left\{ \frac{1}{2}\|y-\theta\|_2^2 + \|\theta\|_{\mathrm{TV},\hat{\omega}} \right\}$$

$\hat{\theta} = \mathrm{prox}_{\|\cdot\|_{\mathrm{TV},\hat{\omega}}}(y)$ [Alaya et al.'15]

1. **set** $k = k_0 = k_- = k_+ \leftarrow 1$; $\theta_{\min} \leftarrow y_1 - \hat{\omega}_2$; $\theta_{\max} \leftarrow y_1 + \hat{\omega}_2$; $u_{\min} \leftarrow \hat{\omega}_2$; $u_{\max} \leftarrow -\hat{\omega}_2$;

2. **if** $k = n$ **then**
  $\hat{\theta}_n \leftarrow \theta_{\min} + u_{\min}$;

3. **if** $y_{k+1} + u_{\min} < \theta_{\min} - \hat{\omega}_{k+2}$ **then** /* negative jump */
  $\hat{\theta}_{k_0} = \cdots = \hat{\theta}_{k_-} \leftarrow \theta_{\min}$; $k = k_0 = k_- = k_+ \leftarrow k_- + 1$;
  $\theta_{\min} \leftarrow y_k - \hat{\omega}_{k+1} + \hat{\omega}_k$; $\theta_{\max} \leftarrow y_k + \hat{\omega}_{k+1} + \hat{\omega}_k$; $u_{\min} \leftarrow \hat{\omega}_{k+1}$; $u_{\max} \leftarrow -\hat{\omega}_{k+1}$;

4. **else if** $y_{k+1} + u_{\max} > \theta_{\max} + \hat{\omega}_{k+2}$ **then** /* positive jump */
  $\hat{\theta}_{k_0} = \ldots = \hat{\theta}_{k_+} \leftarrow \theta_{\max}$; $k = k_0 = k_- = k_+ \leftarrow k_+ + 1$;
  $\theta_{\min} \leftarrow y_k - \hat{\omega}_{k+1} - \hat{\omega}_k$; $\theta_{\max} \leftarrow y_k + \hat{\omega}_{k+1} - \hat{\omega}_k$; $u_{\min} \leftarrow \hat{\omega}_{k+1}$; $u_{\max} \leftarrow -\hat{\omega}_{k+1}$;

5. **else** /* no jump */
  set $k \leftarrow k + 1$; $u_{\min} \leftarrow y_k + \hat{\omega}_{k+1} - \theta_{\min}$; $u_{\max} \leftarrow y_k - \hat{\omega}_{k+1} - \theta_{\max}$;
  if $u_{\min} \geq \hat{\omega}_{k+1}$ **then**
    $\theta_{\min} \leftarrow \theta_{\min} + \frac{u_{\min} - \hat{\omega}_{k+1}}{k - k_0 + 1}$; $u_{\min} \leftarrow \hat{\omega}_{k+1}$; $k_- \leftarrow k$;
  if $u_{\max} \leq -\hat{\omega}_{k+1}$ **then**
    $\theta_{\max} \leftarrow \theta_{\max} + \frac{u_{\max} + \hat{\omega}_{k+1}}{k - k_0 + 1}$; $u_{\max} \leftarrow -\hat{\omega}_{k+1}$; $k_+ \leftarrow k$;

6. **if** $k < n$ **then**
  go to 3.;

7. **if** $u_{\min} < 0$ **then**
  $\hat{\theta}_{k_0} = \cdots = \hat{\theta}_{k_-} \leftarrow \theta_{\min}$; $k = k_0 = k_- \leftarrow k_- + 1$; $\theta_{\min} \leftarrow y_k - \hat{\omega}_{k+1} + \hat{\omega}_k$;
  $u_{\min} \leftarrow \hat{\omega}_{k+1}$; $u_{\max} \leftarrow y_k + \hat{\omega}_k - v_{\max}$; **go to 2.**;

8. **else if** $u_{\max} > 0$ **then**
  $\hat{\theta}_{k_0} = \cdots = \hat{\theta}_{k_+} \leftarrow \theta_{\max}$; $k = k_0 = k_+ \leftarrow k_+ + 1$; $\theta_{\max} \leftarrow y_k + \hat{\omega}_{k+1} - \hat{\omega}_k$;
  $u_{\max} \leftarrow -\hat{\omega}_{k+1}$; $u_{\min} \leftarrow y_k - \hat{\omega}_k - u_{\min}$; **go to 2.**;

9. **else**
  $\hat{\theta}_{k_0} = \cdots = \hat{\theta}_n \leftarrow \theta_{\min} + \frac{u_{\min}}{k - k_0 + 1}$;

Numerical Experiments

Binary Classification

We consider the following datasets for binary classification:

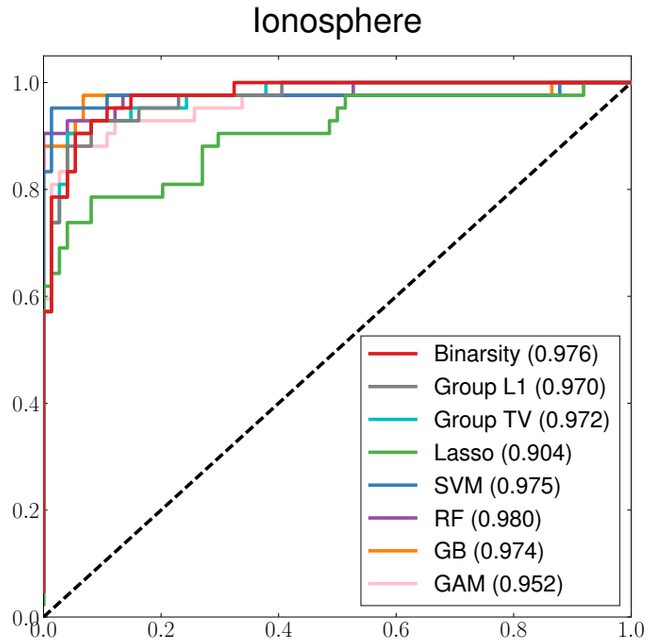
Dataset	#Samples	#Features
Ionosphere	351	34
Churn	3333	21
Default of Credit card	30000	24
Adult	32561	14
Bank Marketing	45211	17
Covertype	550088	10
SUSY	5000000	18
HEPMASS	10500000	28
HIGGS	11000000	24

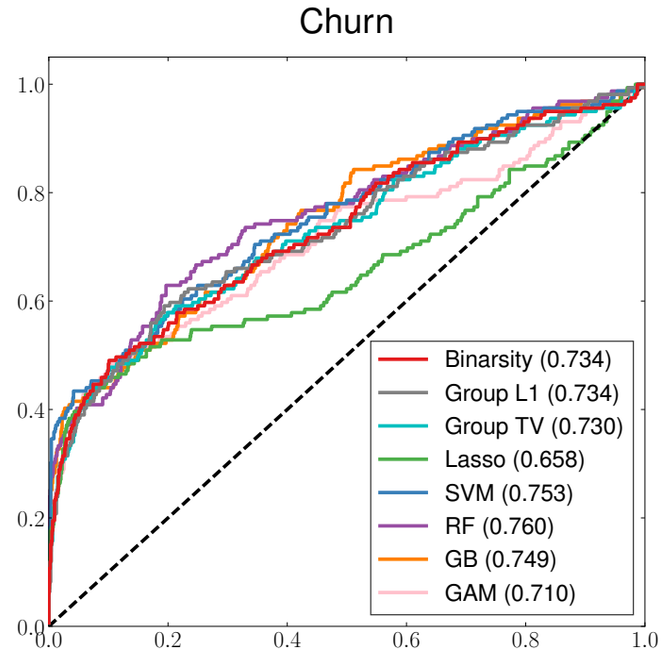
Source: UCI Machine Learning Repository

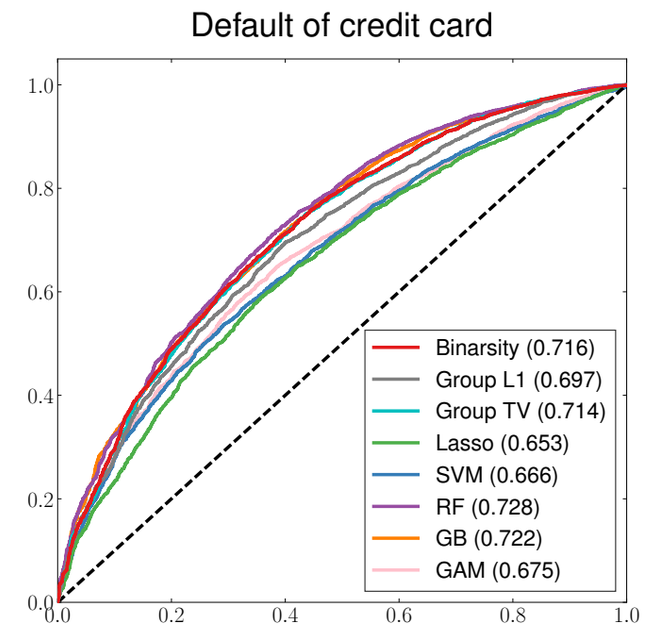
Baselines

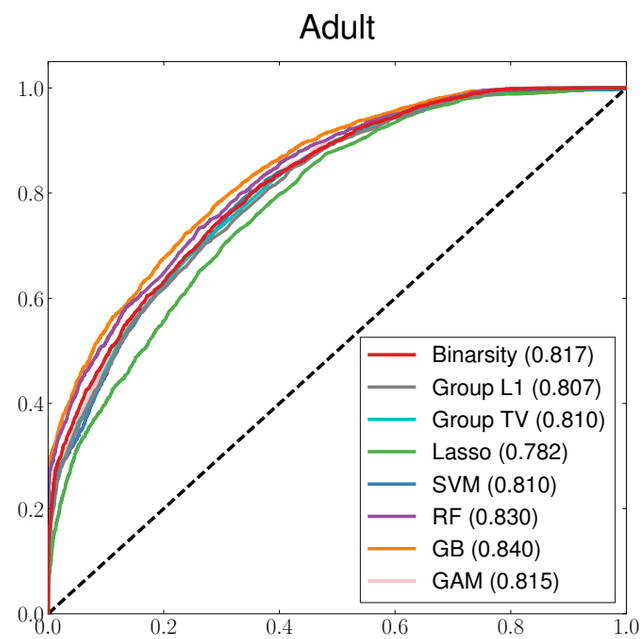
We compare our procedure to the following baseslines

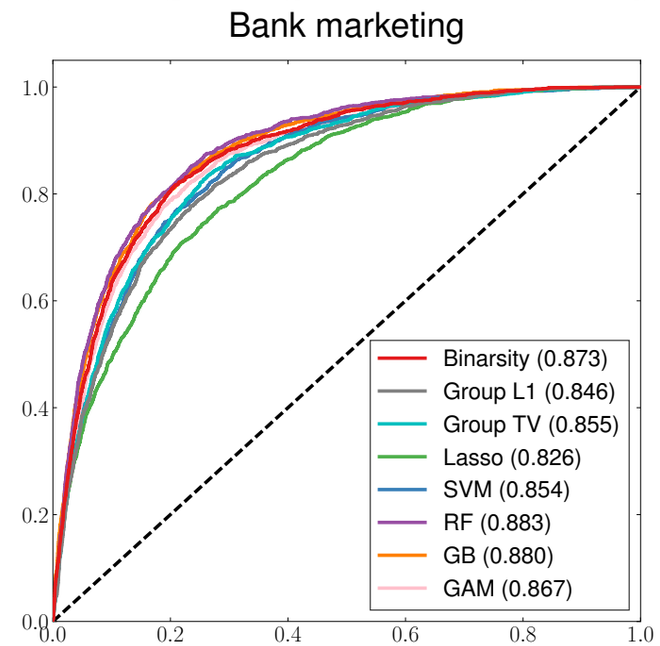
Method	Description
Lasso	Logistic regression with ℓ_1 penalization
Group Lasso	Logistic regression with group ℓ_1 penalization
Group TV	Logistic regression with Group Total-Variation penalization
SVM	Support Vector Machine with Gaussian kernel
GAM	Generalized Additive Model
RF	Random Forest
GB	Gradient Boosting

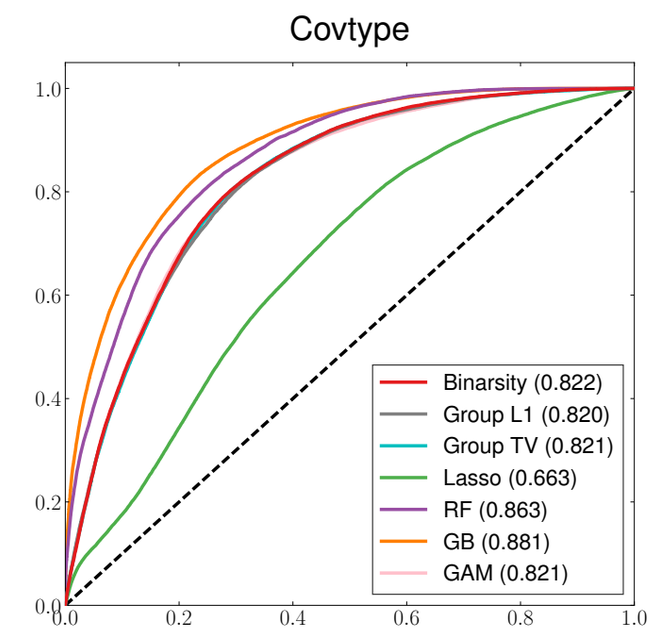

Ionosphere
Binarsity (0.976)
Group L1 (0.970)
Group TV (0.972)
Lasso (0.904)
SVM (0.975)
RF (0.980)
GB (0.974)
GAM (0.952)

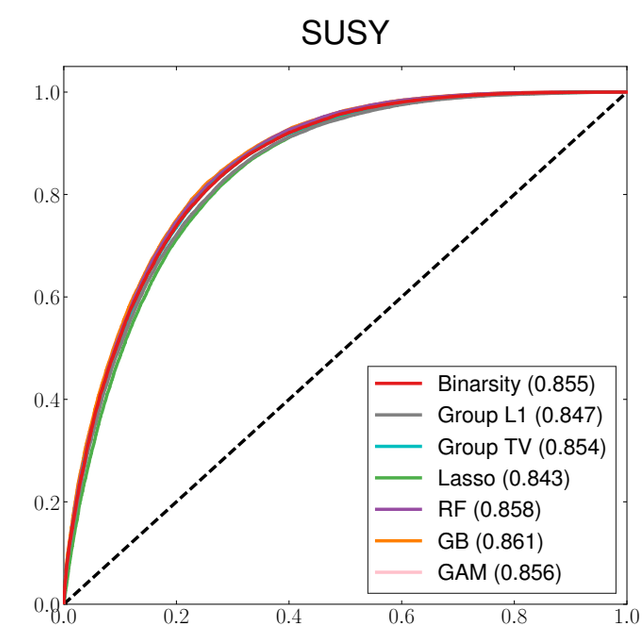

Churn
Binarsity (0.734)
Group L1 (0.734)
Group TV (0.730)
Lasso (0.658)
SVM (0.753)
RF (0.760)
GB (0.749)
GAM (0.710)

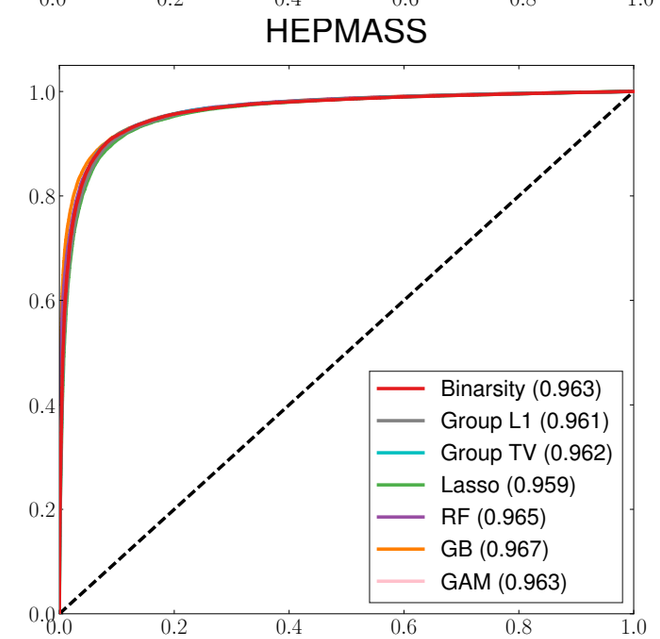

Default of credit card
Binarsity (0.716)
Group L1 (0.697)
Group TV (0.714)
Lasso (0.653)
SVM (0.666)
RF (0.728)
GB (0.722)
GAM (0.675)

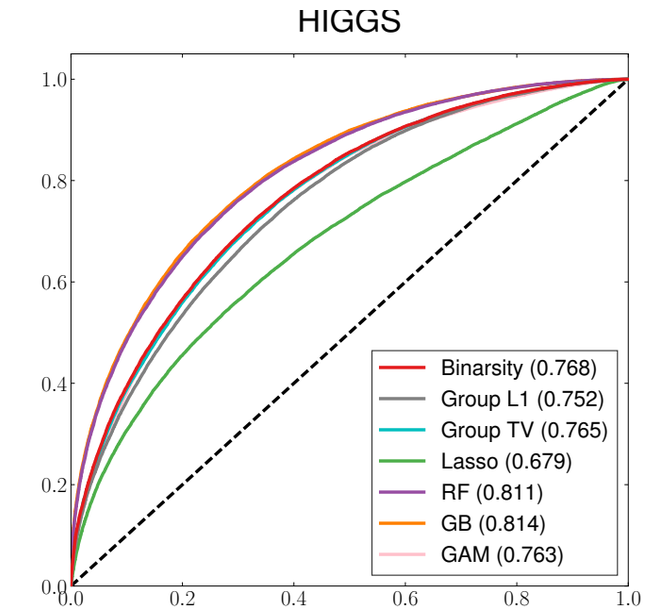

Adult
Binarsity (0.817)
Group L1 (0.807)
Group TV (0.810)
Lasso (0.782)
SVM (0.810)
RF (0.830)
GB (0.840)
GAM (0.815)


Bank marketing
Binarsity (0.873)
Group L1 (0.846)
Group TV (0.855)
Lasso (0.826)
SVM (0.854)
RF (0.883)
GB (0.880)
GAM (0.867)

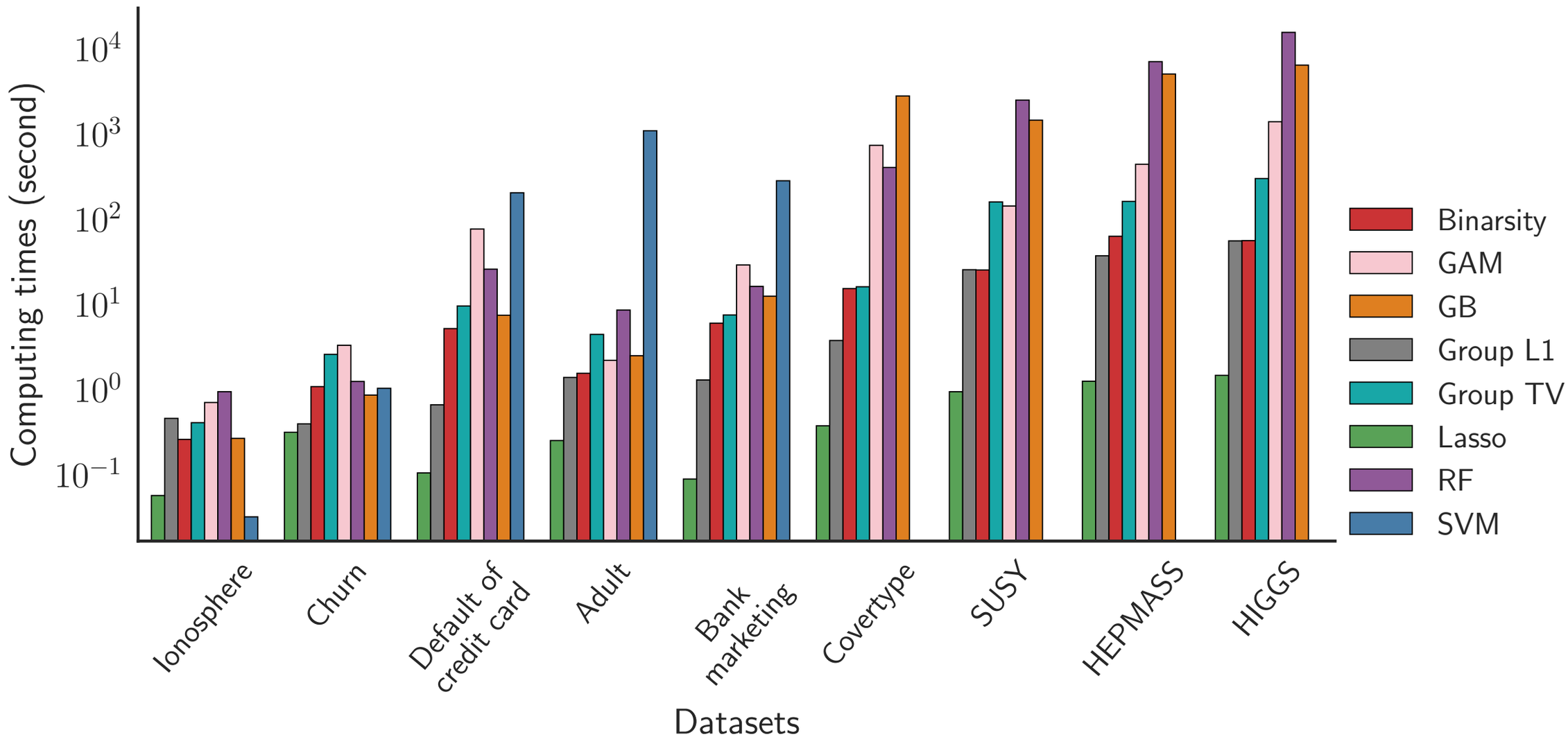

Covtype
Binarsity (0.822)
Group L1 (0.820)
Group TV (0.821)
Lasso (0.663)
RF (0.863)
GB (0.881)
GAM (0.821)


SUSY
Binarsity (0.855)
Group L1 (0.847)
Group TV (0.854)
Lasso (0.843)
RF (0.858)
GB (0.861)
GAM (0.856)


HEPMASS
Binarsity (0.963)
Group L1 (0.961)
Group TV (0.962)
Lasso (0.959)
RF (0.965)
GB (0.967)
GAM (0.963)


HIGGS
Binarsity (0.768)
Group L1 (0.752)
Group TV (0.765)
Lasso (0.679)
RF (0.811)
GB (0.814)
GAM (0.763)

Computing Time Comparisons

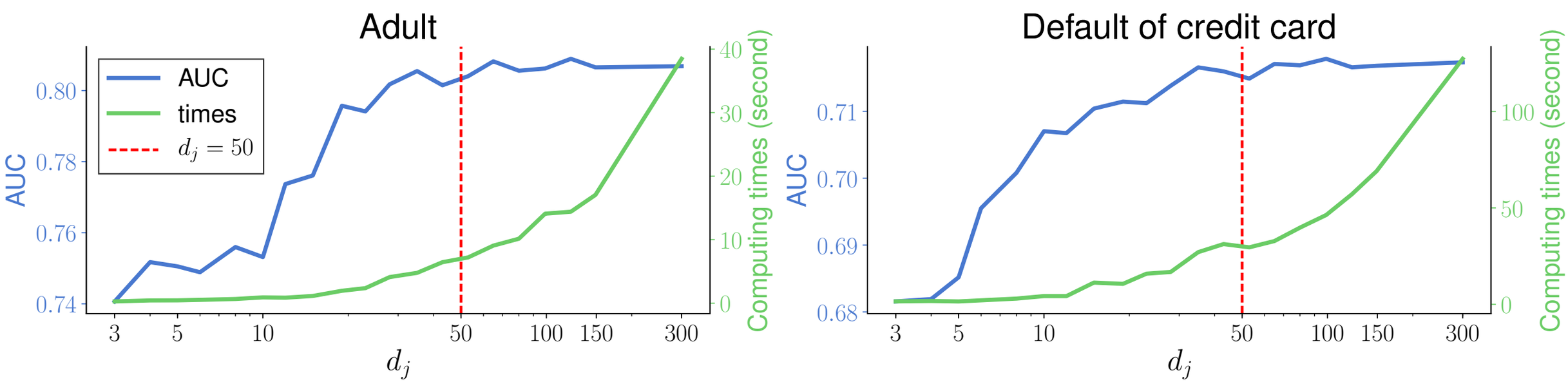


- Binarsity is between 2 and 5 times slower than Lasso but more than 100 times faster than RF and GB on larges datasets like HIGGS.

- For similar performances.

Binarization cuts number

- About the number of bins used in one-hit encoding.
- We can use 50, 100, 200 bins (depending on the number of samples n).

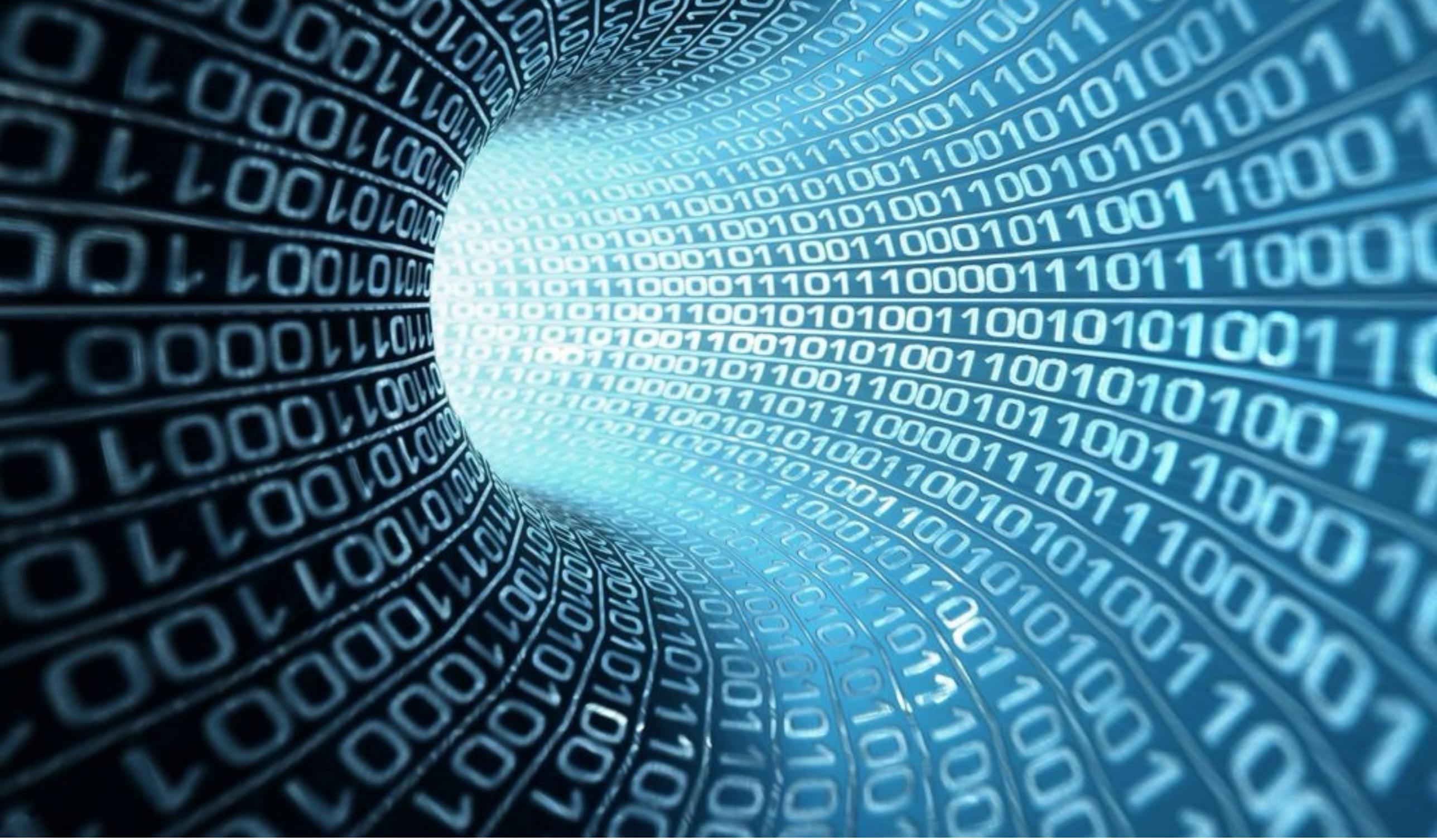


Conclusion

- We introduced the binarsity penalization for one-hot encodings of continuous features

- We illustrated the good statistical properties of binarsity for generalized linear models by proving non-asymptotic oracle inequalities.

- We conducted extensive comparisons of binarsity with state-of-the-art algorithms for binary classification on several standard datasets.



Thank you!